

ON THE NORMAL CLASS OF CURVES AND SURFACES

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ABSTRACT. We are interested in the normal class of an algebraic surface \mathcal{S} of the complex projective space \mathbb{P}^3 , that is the number of normal lines to \mathcal{S} passing through a generic point of \mathbb{P}^3 . Thanks to the notion of normal polar, we state a formula for the normal class valid for a general surface \mathcal{S} . We give a generic result and we illustrate our formula with examples. We complete our work with a generalization of Salmon's formula for the normal class of a Plücker curve to any planar curve with any kind of singularity. This last formula gives directly the normal class of any cylinder and of any surface of revolution of \mathbb{P}^3 .

INTRODUCTION

The notion of normal lines to an hypersurface of an euclidean space is extended here to the complex projective space \mathbb{P}^n . The aim of the present work is to study the **normal class** $c_\nu(\mathcal{S})$ of a surface \mathcal{S} of \mathbb{P}^3 , that is the number of $m \in \mathcal{S}$ such that the projective normal line $\mathcal{N}_m(\mathcal{S})$ to \mathcal{S} at m passing through a generic $m_1 \in \mathbb{P}^3$ (see Section 1 for a precise definition of the projective normal lines and of the normal class of an hypersurface of \mathbb{P}^n). We prove namely the following result.

Theorem 1 ($n=3$). *The normal class of a generic irreducible surface $\mathcal{S} \subset \mathbb{P}^3$ of degree $d_{\mathcal{S}}$ is*

$$c_\nu(\mathcal{S}) = d_{\mathcal{S}}^3 - d_{\mathcal{S}}^2 + d_{\mathcal{S}}.$$

Actually we establish formulas valid for a wider family of surfaces. We illustrate our formulas with the study of the normal class of every quadric and with the computation of the normal class of a cubic surface with singularity E_6 .

The notion of normal polar plays an important role in our study. Given an irreducible surface $\mathcal{S} \subset \mathbb{P}^3$ of degree $d_{\mathcal{S}}$, we extend the definition of the line $\mathcal{N}_m(\mathcal{S})$ to $m \in \mathbb{P}^3$. In our approach, we use the Plücker embedding to identify a line with an element of $\mathbb{G}(1, 3) \subset \mathbb{P}^5$. We then define a rational map $\alpha : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ corresponding to $m \mapsto \mathcal{N}_m(\mathcal{S})$. We write \mathcal{B} for the set of base points of this map α , i.e. the set of points $m \in \mathbb{P}^3$ at which the line $\mathcal{N}_m(\mathcal{S})$ is not well defined. Up to a simplification of α in $\tilde{\alpha} = \frac{\alpha}{H}$ (for some homogeneous polynomial H of degree d_H), this set \mathcal{B} can be assumed to have dimension at most 1. For any $P \in \mathbb{P}^3$, we will introduce the notion of normal polar of \mathcal{S} with respect to P as the set of $m \in \mathbb{P}^3$ such that either $\mathcal{N}_m(\mathcal{S})$ contains P or $m \in \mathcal{B}$. We prove that the normal polar has dimension 1 and degree $\tilde{d}_{\mathcal{S}}^2 - \tilde{d}_{\mathcal{S}} + 1$ for a generic $P \in \mathbb{P}^3$ (with $\tilde{d}_{\mathcal{S}} = d_{\mathcal{S}} - d_H$). If \mathcal{S} has isolated singularities and satisfies some additional conditions, we prove that its normal class $c_\nu(\mathcal{S})$ is equal to $d_{\mathcal{S}}(\tilde{d}_{\mathcal{S}}^2 - \tilde{d}_{\mathcal{S}} + 1)$ minus the intersection multiplicity of \mathcal{S} with its generic normal polars at points $m \in \mathcal{B} \cap \mathcal{S}$. Theorem 1 is a consequence of this general result. Indeed we will see that, for a generic surface $\mathcal{S} \subset \mathbb{P}^3$ of degree $d_{\mathcal{S}}$, we have $\mathcal{S} \cap \mathcal{B} = \emptyset$ and so $\tilde{d}_{\mathcal{S}} = d_{\mathcal{S}}$.

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When the surface is a cylinder or a surface of revolution, its normal class is equal to the normal class of the corresponding basis planar curve. The normal class of any planar curve is given by the simple formula given in Theorem 2 below, that we give for completeness. Let us recall that, when $\mathcal{C} = V(F)$ is an irreducible curve of \mathbb{P}^2 , the evolute of \mathcal{C} is the curve tangent to the family of normal lines to \mathcal{Z} and that the evolute of a line or a circle is reduced to a single point. Hence, except for lines and circles, the normal class of \mathcal{C} is simply the class (with multiplicity) of its evolute. The following result generalizes the result by Salmon [7, p. 137] proved in the case of Plücker curves (planar curves with no worse multiple tangents than ordinary double tangents, no singularities other than ordinary nodes and cusps) to any planar curve (with any type of singularities). We write ℓ_∞ for the line at infinity of \mathbb{P}^2 . We define $I[1 : i : 0]$ and $J[1 : -i : 0]$ in \mathbb{P}^2 . Recall that I and J are the two cyclic points (i.e. the circular points at infinity).

Theorem 2 ($n=2$). *Let $\mathcal{C} = V(F)$ be an irreducible curve of \mathbb{P}^2 of degree $d \geq 2$ with class d^\vee . Then its normal class is*

$$c_\nu(\mathcal{C}) = d + d^\vee - \Omega(\mathcal{C}, \ell_\infty) - \mu_I(\mathcal{C}) - \mu_J(\mathcal{C}),$$

where Ω denotes the sum of the contact numbers between two curves and where $\mu_P(\mathcal{C})$ is the multiplicity of P on \mathcal{C} .

In [3], Fantechi proved that the evolute map is birational from \mathcal{C} to its evolute curve unless if¹ $F_x^2 + F_y^2$ is a square modulo F and that in this latest case the evolute map is $2 : 1$ (if \mathcal{C} is neither a line nor a circle). Therefore, the normal class $c_\nu(\mathcal{C})$ of a planar curve \mathcal{C} corresponds to the class of its evolute unless $F_x^2 + F_y^2$ is a square modulo F and in this last case, the normal class $c_\nu(\mathcal{C})$ of \mathcal{C} corresponds to the class of its evolute times 2 (if \mathcal{C} is neither a line nor a circle).

The notion of focal loci generalizes the notion of evolute to higher dimension [8, 1]. The normal lines of an hypersurface \mathcal{Z} are tangent to the focal loci hypersurface of \mathcal{Z} but of course the normal class of \mathcal{Z} does not correspond anymore (in general) to the class of its focal loci (the normal lines to \mathcal{Z} are contained in but are not equal to the tangent planes of its focal loci).

As usual, we write elements of \mathbb{C}^{n+1} either by (x_1, \dots, x_{n+1}) or by $\begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix}$ (our convention

for line vectors is to write them $(x_1 \cdots x_{n+1})$).

In section 1, we define the projective normal lines, the normal class of an hypersurface \mathcal{Z} of \mathbb{P}^n . In section 2, we study the normal lines to a surface \mathcal{S} of \mathbb{P}^3 , we introduce the rational map α , the normal polars, the set \mathcal{B} , we state a general formula for the normal class of a surface of \mathbb{P}^3 and we prove Theorem 1. In section 3, we apply our results and compute the normal class of every quadric. In section 4, we use our method to compute of the normal class of a cubic surface with singularity E_6 . In section 5, we prove Theorem 2. We end this paper with an appendix on the normal class of two particular kinds of surfaces: the cylinders and the surfaces of revolution.

1. THE PROJECTIVE NORMAL LINES TO AN HYPERSURFACE

Let E_n be an euclidean affine n -space of direction the n -vector space \mathbf{E}_n (endowed with some fix basis). Let $\mathbf{V} := (\mathbf{E}_n \oplus \mathbb{R}) \otimes \mathbb{C}$ (endowed with the induced basis $\mathbf{e}_1, \dots, \mathbf{e}_{n+1}$). We consider the complex projective space $\mathbb{P}^n := \mathbb{P}(\mathbf{V})$ with projective coordinates x_1, \dots, x_{n+1} . We denote by $\mathcal{H}^\infty := V(x_{n+1}) \subset \mathbb{P}^n$ the hyperplane at infinity. Given $\mathcal{Z} = V(F) \neq \mathcal{H}^\infty$ an irreducible hypersurface of \mathbb{P}^n (with $F \in \mathbb{P}(\text{Sym}(\mathbf{V}^\vee)) \cong \mathbb{C}[x_1, \dots, x_{n+1}]$), we consider the rational map $n_{\mathcal{Z}} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ given by $n_{\mathcal{Z}} = [F_{x_1} : \dots : F_{x_n} : 0]$.

¹We write $[x : y : z]$ for the coordinates of $m \in \mathbb{P}^2$ and F_x, F_y, F_z for the partial derivatives of F .

Definition 3. The projective normal line $\mathcal{N}_m \mathcal{Z}$ to \mathcal{Z} at $m \in \mathcal{Z}$ is the line $(m n_{\mathcal{Z}}(m))$ when $n_{\mathcal{Z}}(m)$ is well defined in \mathbb{P}^n and not equal to m .

If F has real coefficients and if $m \in \mathcal{Z} \setminus \mathcal{H}_{\infty}$ has real coordinates $[x_1^{(0)} : \cdots : x_n^{(0)} : 1]$, then $\mathcal{N}_m \mathcal{Z}$ corresponds to the affine normal line of the affine hypersurface $V(F(x_1, \dots, x_n, 1)) \subset E_n$ at the point of coordinates $(x_1^{(0)}, \dots, x_n^{(0)})$. The aim of this work is the study of the notion of normal class.

Definition 4. Let \mathcal{Z} be an irreducible hypersurface of \mathbb{P}^n . The normal class of \mathcal{Z} is the number $c_{\nu}(\mathcal{Z})$ of projective normal lines of \mathcal{Z} through a generic $m_1 \in \mathbb{P}^n$, i.e. the number of $m \in \mathcal{Z}$ such that $\mathcal{N}_m(\mathcal{Z})$ contains m_1 for a generic m_1 .

We will observe that the projective normal lines are preserved by some changes of coordinates in \mathbb{P}^n , called **projective similitudes** of \mathbb{P}^n and defined as follows.

Recall that, for every field \mathbb{k} , $GO(n, \mathbb{k}) = \{A \in GL(n, \mathbb{k}); \exists \lambda \in \mathbb{k}^*, A \cdot {}^t A = \lambda \cdot I_n\}$ is the **orthogonal similitude group** (for the standard products) and that $GOAff(n, \mathbb{k}) = \mathbb{k}^n \rtimes GO(n, \mathbb{k})$ is the **orthogonal similitude affine group**. We have a natural monomorphism of groups $\kappa : Aff(n, \mathbb{R}) = \mathbb{R}^n \rtimes GL(n, \mathbb{R}) \longrightarrow GL(n+1, \mathbb{R})$ given by

$$\kappa(b, A) = \begin{pmatrix} a_{11} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} & b_n \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \quad (1)$$

and, by restriction, $\kappa|_{GOAff(n, \mathbb{R})} : GOAff(n, \mathbb{R}) = \mathbb{R}^n \rtimes GO(n, \mathbb{R}) \longrightarrow GL(n+1, \mathbb{R})$. Analogously we have a natural monomorphism of groups $\kappa' := (\kappa \otimes 1)|_{GOAff(n, \mathbb{C})} : GOAff(n, \mathbb{C}) = \mathbb{C}^n \rtimes GO(n, \mathbb{C}) \longrightarrow GL(n+1, \mathbb{C})$. Composing with the canonical projection $\pi : GL(n+1, \mathbb{C}) \longrightarrow \mathbb{P}(GL(n+1, \mathbb{C}))$ we obtain the **projective complex similitude Group**:

$$\widehat{Sim}_{\mathbb{C}}(n) := (\pi \circ \kappa')(GOAff(n, \mathbb{C})).$$

which acts naturally on \mathbb{P}^n .

Definition 5. An element of $\mathbb{P}(GL(\mathbf{V}))$ corresponding to an element of $\widehat{Sim}_{\mathbb{C}}(n)$ with respect to the basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ is called a **projective similitude of \mathbb{P}^n** .

The set of projective similitudes of \mathbb{P}^n is isomorphic to $\widehat{Sim}_{\mathbb{C}}(n)$.

Lemma 6. The projective similitudes preserve the normal lines.

More precisely, given an hypersurface $\mathcal{Z} = V(F) \subset \mathbb{P}^n$, for any projective similitude φ and any $m \in \mathcal{Z}$ such that $n_{\mathcal{Z}}(m)$ is well defined so is $n_{\varphi(\mathcal{Z})}(\varphi(m))$ and we have $\varphi(n_{\mathcal{Z}}(m)) = n_{\varphi(\mathcal{Z})}(\varphi(m))$. Hence, if $\mathcal{N}_m(\mathcal{Z})$ is well defined, so is $\mathcal{N}_{\varphi(m)}(\varphi(\mathcal{Z}))$ and

$$\varphi(\mathcal{N}_m(\mathcal{Z})) = \mathcal{N}_{\varphi(m)}(\varphi(\mathcal{Z})).$$

Proof. Let $\varphi \in GL(\mathbf{V})$ with matrix $\kappa'(b, A)$ (in the basis $(\mathbf{e}_1, \dots, \mathbf{e}_n)$) of the form (1), with $b \in \mathbb{C}^n$ and $A \in GL(n, \mathbb{C})$ such that $A^t A = \lambda \cdot I_n$. Let φ the projective similitude of \mathbb{P}^n associated to φ . Observe that $\varphi(\mathcal{Z}) = V(G)$ with $G := F \circ \varphi^{-1}$. We write

$$\Delta_{\mathbf{P}} H := x_1^{(0)} H_{x_1} + \cdots + x_{n+1}^{(0)} H_{x_{n+1}}$$

for any $\mathbf{P}(x_1^{(0)}, \dots, x_{n+1}^{(0)}) \in \mathbf{V}$ and $H \in \mathbb{P}(Sym(\mathbf{V}^{\vee}))$. Observe that, for every $\mathbf{B}, \mathbf{P} \in \mathbf{V}$, we have

$$\Delta_{\varphi(\mathbf{B})}(F \circ \varphi^{-1})(\varphi(\mathbf{P})) = \Delta_{\mathbf{B}} F(\mathbf{P}). \quad (2)$$

We consider a nonsingular point $P \in \mathcal{Z}$ such that $\mathcal{T}_P(\mathcal{Z}) \neq \mathcal{H}_\infty$. We set $\mathbf{e}'_i := \varphi(\mathbf{e}_i)$ and $\mathbf{P}' := \varphi(\mathbf{P})$. Due to (2), we obtain that $\mathcal{T}_{\varphi(P)}(\varphi(\mathcal{Z})) \neq \mathcal{H}_\infty$, hence $n_{\varphi(S)}(\varphi(P))$ is also well defined. Moreover we have

$$\begin{aligned} \varphi(\mathbf{n}_S(\mathbf{P})) &= \varphi(\Delta_{\mathbf{e}_1} F(\mathbf{P}) \cdot \mathbf{e}_1 + \cdots + \Delta_{\mathbf{e}_n} F(\mathbf{P}) \cdot \mathbf{e}_n) \\ &= \Delta_{\mathbf{e}'_1} F(\mathbf{P}) \cdot \mathbf{e}'_1 + \cdots + \Delta_{\mathbf{e}'_n} F(\mathbf{P}) \cdot \mathbf{e}'_n \\ &= \Delta_{\mathbf{e}'_1} G(\mathbf{P}') \cdot \mathbf{e}'_1 + \cdots + \Delta_{\mathbf{e}'_n} G(\mathbf{P}') \cdot \mathbf{e}'_n. \end{aligned}$$

Now the first coordinate of this vector is

$$\begin{aligned} [\varphi(\mathbf{n}_S(\mathbf{P}))]_1 &= \Delta_{\mathbf{e}'_1} G(\mathbf{P}') \cdot a_{1,1} + \cdots + \Delta_{\mathbf{e}'_n} G(\mathbf{P}') \cdot a_{1,n} \\ &= \Delta_{a_{1,1} \cdot \mathbf{e}'_1 + \cdots + a_{1,n} \cdot \mathbf{e}'_n} G(\mathbf{P}') \\ &= \Delta_{\varphi(a_{1,1} \cdot \mathbf{e}_1 + \cdots + a_{1,n} \cdot \mathbf{e}_n)} G(\mathbf{P}') \\ &= \Delta_{(A(a_{1,1}, \dots, a_{1,n}), 0)} G(\mathbf{P}') \\ &= \Delta_{(A^t A \mathbf{e}_1), 0)} G(\mathbf{P}') \\ &= \Delta_{\lambda \cdot \mathbf{e}_1} G(\mathbf{P}') = \lambda \cdot \Delta_{\mathbf{e}_1} G(\mathbf{P}'). \end{aligned}$$

Treating analogously the other coordinates we obtain that

$$\begin{aligned} \varphi(\mathbf{n}_S(\mathbf{P})) &= \lambda \cdot (\Delta_{\mathbf{e}_1} G(\mathbf{P}') + \cdots + \Delta_{\mathbf{e}_n} G(\mathbf{P}')) \\ &= \lambda \cdot \mathbf{n}_{\varphi(S)}(\mathbf{P}') \end{aligned}$$

and so $\varphi(n_S(P)) = n_{\varphi(S)}(\varphi(P))$. \square

Remark 7. *It is worth noting that, due to this lemma, the normal class of surfaces of \mathbb{P}^3 is invariant by projective similitudes of \mathbb{P}^3 .*

2. STUDY OF THE NORMAL CLASS OF SURFACES

Let \mathbf{V} be a four dimensional complex linear space (endowed with a basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$) and $\mathbb{P}^3 := \mathbb{P}(\mathbf{V})$ with projective coordinates x, y, z, t . In \mathbb{P}^3 , we consider an irreducible surface $\mathcal{S} = V(F)$ with homogeneous $F \in \mathbb{C}[x, y, z, t]$ of degree d_S . Recall that we write $\mathcal{H}^\infty = V(t)$ for the plane at infinity. For any nonsingular $m \in \mathcal{S}$ (with coordinates $\mathbf{m} = (x, y, z, t) \in \mathbb{C}^4$), we write $\mathcal{T}_m \mathcal{S}$ for the tangent plane to \mathcal{S} at m . If $\mathcal{T}_m \mathcal{S} \neq \mathcal{H}^\infty$, then

$$n_S(m) = [F_x(\mathbf{m}) : F_y(\mathbf{m}) : F_z(\mathbf{m}) : 0] \quad (3)$$

is well defined in \mathbb{P}^3 . If moreover $n_S(m) \neq m$, we define the projective normal line $\mathcal{N}_m \mathcal{S}$ to \mathcal{S} at m as the line $(m n_S(m))$. We will associate to a generic $m \in \mathcal{S}$ a system of equations of $\mathcal{N}_m \mathcal{S}$. This will be done thanks to the Plücker embedding. Observe that n_S given by (3) defines a rational map $n_S : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$.

2.1. Plücker embedding. Let $W \equiv \mathbf{V}^\vee$ be the set of homogeneous polynomials of degree one in $\mathbb{C}[x, y, z, t]$, endowed with $(e_1^*, e_2^*, e_3^*, e_4^*)$ the dual basis of $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$. We consider the duality $\delta : W \dashrightarrow \mathbb{P}^3$ given by $\delta(ae_1^* + be_2^* + ce_3^* + de_4^*) = [a : b : c : d]$. We define $\delta^{(4)} : W^4 \dashrightarrow (\mathbb{P}^3)^4$ by

$$\delta^{(4)}(l_1, l_2, l_3, l_4) = (\delta(l_1), \delta(l_2), \delta(l_3), \delta(l_4)).$$

We consider now $\Lambda^3 : \mathbf{V} \times \mathbf{V} \times W^4 \rightarrow W^4$ given by

$$\Lambda^3(\mathbf{u}, \mathbf{v}, \ell) = (\widetilde{M}_1, -\widetilde{M}_2, \widetilde{M}_3, -\widetilde{M}_4),$$

where \widetilde{M}_j is the determinant of the minor obtained from $M = (\mathbf{u} \ \mathbf{v} \ \ell)$ by deleting the j -th row of M .

Let Δ be the diagonal of $\mathbb{P}^3 \times \mathbb{P}^3$. We define $\mu : (\mathbb{P}^3 \times \mathbb{P}^3) \setminus \Delta \rightarrow (\mathbb{P}^3)^4$ by

$$\mu(\mathbf{u}, \mathbf{v}) = \delta^{(4)}(\Lambda^3(\mathbf{u} \mathbf{v} L)) \text{ where } L := \begin{pmatrix} e_1^* \\ e_2^* \\ e_3^* \\ e_4^* \end{pmatrix}.$$

Observe that when $u \neq v$, $\Lambda^3(\mathbf{u} \mathbf{v} L)$ gives four equations in $\mathbb{C}[x, y, z, t]$ of the line (uv) , i.e.

$$(uv) = \cap_{i=1}^4 V(\langle \mu_i(u, v), \cdot \rangle) \subset \mathbb{P}^3,$$

where $\mu(u, v) = (\mu_1(u, v), \mu_2(u, v), \mu_3(u, v), \mu_4(u, v))$ and where we write $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^4 u_i v_i$ for the complexification of the usual scalar product on \mathbb{R}^4 to \mathbf{V} .

We will express ν in terms of the **Plücker embedding** $(\mathbb{P}^3 \times \mathbb{P}^3) \setminus \Delta \xrightarrow{Pl} \mathbb{P}^5 = \mathbb{P}(\Lambda^2 \mathbb{C}^4)$ defined by

$$Pl(u, v) = [\mathbf{u} \wedge \mathbf{v}] = \begin{bmatrix} u_1 v_2 - v_1 u_2 \\ u_1 v_3 - v_1 u_3 \\ u_1 v_4 - v_1 u_4 \\ u_2 v_3 - v_2 u_3 \\ u_2 v_4 - v_2 u_4 \\ u_3 v_4 - v_3 u_4 \end{bmatrix}.$$

The image of the Plücker embedding is the quadric hypersurface (see [2])

$$\mathbb{G}(1, 3) := Pl((\mathbb{P}^3)^2 \setminus \Delta) = V(w_1 w_6 - w_2 w_5 + w_3 w_4) \subset \mathbb{P}^5.$$

We have a first commutative diagram (for $i = 1, \dots, 4$)

$$\begin{array}{ccccc} & & (\mathbb{P}^3)^4 & \xrightarrow{pr_i} & \mathbb{P}^3 \\ & \nearrow \mu & \uparrow \Psi & \nearrow \psi_i & \\ (\mathbb{P}^3 \times \mathbb{P}^3) \setminus \Delta & \xrightarrow{Pl} & \mathbb{P}^5 = \mathbb{P}(\Lambda^2 \mathbf{V}) & & \end{array} \quad (4)$$

where pr_i is the canonical projection (on the i -th coordinate) and where $\Psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ is given on coordinates by

$$\Psi(w_1, w_2, w_3, w_4, w_5, w_6) = \left(\begin{pmatrix} 0 \\ w_6 \\ -w_5 \\ w_4 \end{pmatrix}, \begin{pmatrix} -w_6 \\ 0 \\ w_3 \\ -w_2 \end{pmatrix}, \begin{pmatrix} w_5 \\ -w_3 \\ 0 \\ w_1 \end{pmatrix}, \begin{pmatrix} -w_4 \\ w_2 \\ -w_1 \\ 0 \end{pmatrix} \right) \in \mathbf{V}^4.$$

Observe that the images of Ψ correspond to antisymmetric matrices (by identifying $(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{w}') \in \mathbf{V}^4$ with the matrix $M_\psi := (\mathbf{u} \mathbf{v} \mathbf{w} \mathbf{w}')$).

2.2. Equations of projective normal lines. Now we come back to normal lines. Let $j : \mathbb{P}^3 \rightarrow \mathbb{P}^3 \times \mathbb{P}^3$ be the morphism defined by $j(m) = (m, n_S(m))$. We define $\alpha : \mathbb{P}^3 \dashrightarrow \mathbb{P}^5$ by

$$\alpha := Pl \circ j.$$

We observe that the set of base points of α is $\mathcal{B} := j^{-1}(\Delta)$, i.e.

$$\mathcal{B} = \{m \in \mathbb{P}^3; \text{rank}(\mathbf{m}, \mathbf{n}_S(\mathbf{m})) < 2\}.$$

We define also $\nu := \mu \circ j$ which maps any $m \in \mathcal{S} \setminus \Delta$ to the coordinates of a system of four equations of the normal line $\mathcal{N}_m \mathcal{S}$ to \mathcal{S} at m . We deduce from (4) a second commutative diagram

$$\begin{array}{ccc} & (\mathbb{P}^3)^4 & \xrightarrow{pr_i} \mathbb{P}^3 \\ & \nearrow \nu & \uparrow \Psi \quad \nearrow \psi_i \\ \mathbb{P}^3 \setminus \mathcal{B} & \xrightarrow{\alpha} & \mathbb{P}^5 \end{array} \quad (5)$$

Observe that $\alpha(x, y, z, t) = \begin{pmatrix} xF_y - yF_x \\ xF_z - zF_x \\ -tF_x \\ yF_z - zF_y \\ -tF_y \\ -tF_z \end{pmatrix}$ and so

$$\nu(x, y, z, t) = \left(\begin{bmatrix} 0 \\ -tF_z \\ tF_y \\ yF_z - zF_y \end{bmatrix}, \begin{bmatrix} tF_z \\ 0 \\ -tF_x \\ zF_x - xF_z \end{bmatrix}, \begin{bmatrix} -tF_y \\ tF_x \\ 0 \\ xF_y - yF_x \end{bmatrix}, \begin{bmatrix} zF_y - yF_z \\ xF_z - zF_x \\ yF_x - xF_y \\ 0 \end{bmatrix} \right).$$

2.3. The normal polars of \mathcal{S} . Let m_1 be a generic point of \mathbb{P}^3 . By construction, for every $m[x : y : z : t] \in \mathcal{S} \setminus \mathcal{B}$, we have $\mathcal{N}_m \mathcal{S} = \bigcap_{i=1}^4 V(\langle \nu_i(\mathbf{m}), \cdot \rangle)$ where $\nu_i := pr_i \circ \nu$. For every $A \in \mathbf{V}^\vee$, we define **the normal polar** $\mathcal{P}_{A, \mathcal{S}}$ of \mathcal{S} with respect to A by

$$\mathcal{P}_{A, \mathcal{S}} := \bigcap_{i=1}^4 V(A \circ \nu_i) = \mathcal{B} \cup \left(\bigcap_{i=1}^4 \nu_i^{-1} \mathcal{H}_A \right),$$

where $\mathcal{H}_A := V(A) \subset \mathbb{P}^3$. For every $m \in \mathbb{P}^3$ and every $A \in \mathbf{V}^\vee$, we have

$$m \in \mathcal{P}_{A, \mathcal{S}} \Leftrightarrow m \in \mathcal{B} \text{ or } \delta(A) \in \mathcal{N}_m(\mathcal{S}),$$

extending the definition of $\mathcal{N}_m(\mathcal{S})$ from $m \in \mathcal{S}$ to $m \in \mathbb{P}^3$. From the geometric point of view, due to (5), we observe that

$$\mathcal{P}_{A, \mathcal{S}} = \bigcap_{i=1}^4 V((A \circ \psi_i) \circ \alpha) = \mathcal{B} \cup \left(\bigcap_{i=1}^4 \alpha^{-1} \mathcal{H}_{A, i} \right),$$

where $\mathcal{H}_{A, i}$ is the hyperplane of \mathbb{P}^5 given by

$$\mathcal{H}_{A, i} := V(A \circ \psi_i) = \text{Base}(\Psi_i) \cup \Psi_i^{-1}(\mathcal{H}).$$

Lemma 8. *For every $A \in \mathbf{V}^\vee \setminus \{0\}$, the set $\bigcap_{i=1}^4 \mathcal{H}_{A, i}$ is a plane of \mathbb{P}^5 contained in $\mathbb{G}(1, 3)$.*

Proof. Let $A = ae_1^* + be_2^* + ce_3^*z + de_4^*$ be an element of \mathbf{V}^\vee and write $B := \delta(A) = [a : b : c : d]$ and $\mathbf{B} := (a, b, c, d) \in \mathbf{V}$. Assume for example $d \neq 0$ (the proof being analogous when $a \neq 0$, $b \neq 0$ or $c \neq 0$, for symmetry reason). Observe that $a \cdot A \circ \psi_1 + b \cdot A \circ \psi_2 + c \cdot A \circ \psi_3 + d \cdot A \circ \psi_4 = ({}^t \mathbf{B}) \cdot M_\psi \cdot \mathbf{B} = 0$ since M_ψ is antisymmetric. Hence

$$\bigcap_{i=1}^4 \mathcal{H}_{A, i} = \bigcap_{i=1}^3 \mathcal{H}_{A, i}.$$

Recall that $A \circ \psi_1 = bw_6 - cw_5 + dw_4$, $A \circ \psi_2 = -aw_6 + cw_3 - dw_2$ and $A \circ \psi_3 = aw_5 - bw_3 + dw_1$. Since $d \neq 0$, these three linear equations are linearly independent and so $\bigcap_{i=1}^3 \mathcal{H}_{A, i}$ is a plane of \mathbb{P}^5 . Moreover, we observe that

$$w_3 \cdot (A \circ \psi_1) + w_5 \cdot (A \circ \psi_1) + w_6 \cdot (A \circ \psi_3) = d(w_1w_6 - w_2w_5 + w_3w_4)$$

and so, since $d \neq 0$, it follows that $\bigcap_{i=1}^3 \mathcal{H}_{A,i} \subset V(w_1 w_6 - w_2 w_5 + w_3 w_4) = \mathbb{G}(1, 3)$. \square

Due to the proof of the previous lemma that, if $d \neq 0$, we have

$$\mathcal{P}_{A,S} = \bigcap_{i=1}^3 V(A \circ \nu_i).$$

Proposition 9. *Assume $d_S \geq 2$, $\dim \mathcal{B} \leq 1$, that $F_x F_y F_z \neq 0$ in $\mathbb{C}[x, y, z, t]$. For a generic $A \in \mathbf{V}^\vee$, we have $\dim \mathcal{P}_{A,S} = 1$ and $\deg \mathcal{P}_{A,S} = d_S^2 - d_S + 1$.*

The assumptions of this proposition are valid for a generic surface and are discussed in Remark 10 and in Proposition 11.

Proof of Proposition 9. Let us notice that, since $\dim \mathcal{B} \leq 1$, at least one of the following varieties has dimension less than one : $V(tF_x, tF_y, F_z)$, $V(tF_x, tF_z, F_y)$ and $V(tF_y, tF_z, F_x)$.

Indeed, $\dim V(tF_x, tF_y, F_z) \geq 2$ implies the existence of an irreducible $G \in \mathbb{C}[c, y, z, t]$ dividing tF_x, tF_y and F_z . G divides F_x, F_y and F_z would contradict $\dim \mathcal{B} \leq 1$. So $\dim V(tF_x, tF_y, F_z) \geq 2$ implies that t divides F_z . Hence if the three varieties above were of dimension larger than 1, t would divide F_x, F_y and F_z , which contradicts $\dim \mathcal{B} \leq 1$.

Assume that $\dim V(tF_x, tF_y, F_z) \leq 1$ (the proof being analogous in the other cases). For $A \in \mathbf{V}^\vee$, we write $[a : b : c : d] := \delta(A)$. Assume that $d \neq 0$. Then $\mathcal{P}_{A,S} = \bigcap_{i=1}^3 V(A \circ \nu_i)$ with

$$\begin{aligned} A \circ \nu_1 &= t(cF_y - bF_z) + d(yF_z - zF_y) = F_z(dy - bt) + F_y(ct - dz), \\ A \circ \nu_2 &= t(aF_z - cF_x) + d(zF_x - xF_z) = F_z(at - dx) + F_x(dz - ct), \\ A \circ \nu_3 &= t(bF_x - aF_y) + d(xF_y - yF_x) = F_y(dx - at) + F_x(tb - dy). \end{aligned} \quad (6)$$

We notice that

$$(ct - dz) \cdot A \circ \nu_3 = (dx - at) \cdot A \circ \nu_1 + (dy - bt) \cdot A \circ \nu_2. \quad (7)$$

Let us write $\mathcal{H}_{c,d}$ for the plane $V(ct - dz) \subset \mathbb{P}^3$. Hence, due to (6), we have

$$\mathcal{P}_{A,S} \setminus \mathcal{H}_{c,d} = V(A \circ \nu_1, A \circ \nu_2) \setminus \mathcal{H}_{c,d} \quad (8)$$

and

$$\mathcal{H}_{c,d} \cap V(A \circ \nu_1, A \circ \nu_2) = (\mathcal{H}_{c,d} \cap V(F_z)) \cup \{\delta(A)\}. \quad (9)$$

Observe that $\dim \mathcal{B} \leq 1$ implies that $\dim \mathcal{P}_{A,S} = 1$ for a generic $A \in \mathbf{V}^\vee$. Since $\dim V(tF_x, tF_y, F_z) \leq 1$ and $F_z \neq 0$, for a generic $A \in \mathbf{V}^\vee$, we have $\dim(F_z, A \circ \nu_3) = 1$ and $\#(\mathcal{H}_{c,d} \cap V(F_z, A \circ \nu_3)) < \infty$, i.e. $\#(\mathcal{H}_{c,d} \cap \mathcal{P}_{A,S}) < \infty$. Due to (8) and (9), $\dim \mathcal{P}_{A,S} = 1$ and $\#(\mathcal{H}_{c,d} \cap \mathcal{P}_{A,S}) < \infty$ implies that $\dim(V(A \circ \nu_1, A \circ \nu_2)) = 1$.

Let $A \in \mathbf{V}^\vee$ be such that $\dim \mathcal{P}_{A,S} = 1$ and $\#(\mathcal{H}_{c,d} \cap V(F_z, A \circ \nu_3)) < \infty$. Observe that $\dim(V(F_z) \cap \mathcal{H}_{c,d}) = 1$. Let $\mathcal{H}_0 \subset \mathbb{P}^3$ be a generic plane such that $\mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap \mathcal{P}_{A,S} = \emptyset$ and $\dim(\mathcal{H}_0 \cap V(A \circ \nu_1, A \circ \nu_2)) = 0$. Hence \mathcal{H}_0 does not contain $\delta(A)$. Due to (7) and to the Bezout theorem, we have

$$\begin{aligned} \deg \mathcal{P}_{A,S} &= \sum_{P \in \mathcal{H}_0 \cap \mathcal{P}_{A,S}} i_P(\mathcal{H}_0, \mathcal{P}_{A,S}) \\ &= \sum_{P \in (\mathcal{H}_0 \cap \mathcal{P}_{A,S}) \setminus \mathcal{H}_{c,d}} i_P(\mathcal{H}_0, V(A \circ \nu_1, A \circ \nu_2)) \end{aligned}$$

since $\mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap \mathcal{P}_{A,S} = \emptyset$ and due to (7). Using (8), we obtain

$$\begin{aligned}
\deg \mathcal{P}_{A,S} &= \sum_{P \in \mathcal{H}_0 \cap V(A \circ \nu_1, A \circ \nu_2)} i_P(\mathcal{H}_0, V(A \circ \nu_1, A \circ \nu_2)) - \sum_{P \in \mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(A \circ \nu_1, A \circ \nu_2)} i_P(\mathcal{H}_0, V(A \circ \nu_1, A \circ \nu_2)) \\
&= d_S^2 - \sum_{P \in \mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(A \circ \nu_1, A \circ \nu_2)} i_P(\mathcal{H}_0, V(A \circ \nu_1, A \circ \nu_2, (ct - dz)A \circ \nu_3)) \\
&= d_S^2 - \sum_{P \in \mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(A \circ \nu_1, A \circ \nu_2)} i_P(\mathcal{H}_0, \mathcal{H}_{c,d}, V(A \circ \nu_1, A \circ \nu_2)) \tag{10}
\end{aligned}$$

due to (7) and since $\mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap \mathcal{P}_{A,S} = \emptyset$. Moreover, due to (6), $\mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(A \circ \nu_1, A \circ \nu_2) = \mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(F_z)$ and, for every $P \in \mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(A \circ \nu_1, A \circ \nu_2)$, we have

$$\begin{aligned}
i_P(\mathcal{H}_0, \mathcal{H}_{c,d}, V(A \circ \nu_1, A \circ \nu_2)) &= i_P(\mathcal{H}_0, \mathcal{H}_{c,d}, V(F_z(dy - bt), F_z(at - dx))) \\
&= i_P(\mathcal{H}_0, \mathcal{H}_{c,d}, V(F_z))
\end{aligned}$$

since $\delta(A) \notin \mathcal{H}_0$ and so

$$\deg \mathcal{P}_{A,S} = d_S^2 - \sum_{P \in \mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(F_z)} i_P(\mathcal{H}_0, V(F_z), \mathcal{H}_{c,d}) = d_S^2 - (d_S - 1).$$

□

Remark 10. Observe that, in the computation of the normal class, up to a change of variable by an element of $\widehat{\text{Sim}}_{\mathbb{C}}(3)$, one can always suppose that $F_x F_y F_z = 0$.

Moreover, if $F_z = 0$ in $\mathbb{C}[x, y, z, t]$, then \mathcal{S} is a cylinder of axis $V(x, y)$ and

$$c_\nu(\mathcal{S}) = c_\nu(\mathcal{C}),$$

where $\mathcal{C} := V(G) \subset \mathbb{P}^2$ (with $G(x, y, z) = F(x, y, 0, z)$) is the base-curve of the cylinder \mathcal{S} and Theorem 2 applies to \mathcal{C} (see Proposition 22).

Proposition 11. If $\dim \mathcal{B} = 2$, then the two dimensional part of \mathcal{B} is $V(H) \subset \mathbb{P}^3$ for some homogeneous polynomial $H \in \mathbb{C}[x, y, z, t]$ of degree d_H . We write $\alpha = H \cdot \tilde{\alpha}$. Observe that the rational map $\tilde{\alpha} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ associated to $\tilde{\alpha}$ coincide with α on $\mathbb{P}^3 \setminus V(H)$ and that the set $\tilde{\mathcal{B}}$ of its base points has dimension at most 1. We then adapt our study by replacing ν by $\tilde{\nu}$ associated to $\tilde{\nu}_i := \frac{\nu_i}{H}$ and by defining the corresponding polar $\tilde{\mathcal{P}}_{A,S}$. Assume moreover that $F_x F_y F_z \neq 0$ in $\mathbb{C}[x, y, z, t]$. Then, we have

$$\deg \tilde{\mathcal{P}}_{A,S} = (d_S - d_H)^2 - d_S + d_H + 1.$$

Proof. We explain how the proof of Proposition 9 can be adapted to prove this remark. We have $\tilde{\mathcal{P}}_{A,S} = V(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$, with $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \in \mathbb{C}[x, y, z, t]$ such that $A \circ \nu_i = H \cdot \tilde{A}_i$ for $i \in \{1, 2, 3\}$. Observe that, due to (7), we have

$$(ct - dz) \cdot \tilde{A}_3 = (dx - at) \cdot \tilde{A}_1 + (dy - bt) \cdot \tilde{A}_2. \tag{11}$$

Hence we have

$$\tilde{\mathcal{P}}_{A,S} \setminus \mathcal{H}_{c,d} = V(\tilde{A}_1, \tilde{A}_2) \setminus \mathcal{H}_{c,d}. \tag{12}$$

We distinguish two cases.

- Assume first that H divides F_x, F_y and F_z . Then, we define $\tilde{F}_x, \tilde{F}_y, \tilde{F}_z \in \mathbb{C}[x, y, z, t]$ such that $F_x = H \cdot \tilde{F}_x$, $F_y = H \cdot \tilde{F}_y$ and $F_z := H \cdot \tilde{F}_z$. We have

$$\begin{aligned}
\tilde{A}_1 &= \tilde{F}_z(dy - bt) + \tilde{F}_y(ct - dz), \\
\tilde{A}_2 &= \tilde{F}_z(at - dx) + \tilde{F}_x(dz - ct), \\
\tilde{A}_3 &= \tilde{F}_y(dx - at) + \tilde{F}_x(tb - dy)
\end{aligned} \tag{13}$$

and so

$$\mathcal{H}_{c,d} \cap V(\tilde{A}_1, \tilde{A}_2) = (\mathcal{H}_{c,d} \cap V(\tilde{F}_z)) \cup \{\delta(A)\}. \quad (14)$$

As in the proof of Proposition 9, we can assume that $\dim V(t\tilde{F}_x, t\tilde{F}_y, \tilde{F}_z) \leq 1$ and we take a generic $A \in \mathbf{V}^\vee$ such that $\dim \tilde{\mathcal{P}}_{A,S} = 1$ and $\#(\mathcal{H}_{c,d} \cap \tilde{\mathcal{P}}_{A,S}) < \infty$. In particular, we have $\dim V(\tilde{A}_1, \tilde{A}_2) = 1$ and $\dim(V(\tilde{F}_z) \cap \mathcal{H}_{c,d}) = 1$. We consider a generic plane $\mathcal{H}_0 \subset \mathbb{P}^3$ such that $\mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap \mathcal{P}_{A,S} = \emptyset$ and $\dim(\mathcal{H}_0 \cap V(\tilde{A}_1, \tilde{A}_2)) = 0$. Following the proof of (10), we obtain that, for a generic plane $\mathcal{H}_0 \subset \mathbb{P}^3$

$$\begin{aligned} \deg \tilde{\mathcal{P}}_{A,S} &= \deg(\tilde{A}_1)^2 - \sum_P i_P(\mathcal{H}_0, \mathcal{H}_{c,d}, V(\tilde{A}_1, \tilde{A}_2)) \\ &= \deg(\tilde{A}_1)^2 - \sum_P i_P(\mathcal{H}_0, \mathcal{H}_{c,d}, V(\tilde{F}_z)) = \deg(\tilde{A}_1)^2 - \deg \tilde{F}_z, \end{aligned} \quad (15)$$

due to (14), to (13) and since $\delta(A) \notin \mathcal{H}_0$.

- Assume now that H does not divide F_x or does not divide F_y or does not divide F_z . Due to the expression of α , H divides $tF_x, tF_y, tF_z, xF_y - yF_x, xF_z - zF_x, yF_z - zF_y$. Hence t divides $xF_y - yF_x, xF_z - zF_x, yF_z - zF_y$. This implies that d_S is even and the existence of a non null polynomial $G \in \mathbb{C}[r]$ of degree $d_S/2$ and of an homogeneous polynomial $K \in \mathbb{C}[x, y, z, t]$ of degree $d_S - 1$ such that

$$F(x, y, z, t) = G(x^2 + y^2 + z^2) + tK(x, y, z, t). \quad (16)$$

Hence t divides H , we set $H_1 \in \mathbb{C}[x, y, z, t]$ such that $H = tH_1$ and we define $\tilde{F}_x, \tilde{F}_y, \tilde{F}_z \in \mathbb{C}[x, y, z, t]$ such that $F_x = H_1 \cdot \tilde{F}_x$, $F_y = H_1 \cdot \tilde{F}_y$ and $F_z := H_1 \cdot \tilde{F}_z$. Observe that

$$\begin{aligned} \tilde{A}_1 &= c\tilde{F}_y - b\tilde{F}_z + d\tilde{G}_{y,z} \text{ and } t\tilde{A}_1 = \tilde{F}_z(dy - bt) + \tilde{F}_y(ct - dz), \\ \tilde{A}_2 &= a\tilde{F}_z - c\tilde{F}_x - d\tilde{G}_{x,z} \text{ and } t\tilde{A}_2 = \tilde{F}_z(at - dx) + \tilde{F}_x(dz - ct), \\ \tilde{A}_3 &= b\tilde{F}_x - a\tilde{F}_y + d\tilde{G}_{x,y} \text{ and } t\tilde{A}_3 = \tilde{F}_y(dx - at) + \tilde{F}_x(tb - dy). \end{aligned} \quad (17)$$

with $\tilde{G}_{x,y}, \tilde{G}_{x,z}, \tilde{G}_{y,z} \in \mathbb{C}[x, y, z, t]$ such that

$$\begin{aligned} xF_y - yF_x &= H \cdot \tilde{G}_{x,y} = H_1 \cdot (x\tilde{F}_y - y\tilde{F}_x) \\ xF_z - zF_x &= H \cdot \tilde{G}_{x,z} = H_1 \cdot (x\tilde{F}_z - z\tilde{F}_x) \\ yF_z - zF_y &= H \cdot \tilde{G}_{y,z} = H_1 \cdot (y\tilde{F}_z - z\tilde{F}_y). \end{aligned}$$

Observe that $\dim V(\tilde{F}_x, \tilde{F}_y, \tilde{F}_z) \leq 1$. Moreover $\tilde{F}_z \neq 0$. Due to (16), it will be useful to notice that $V(x, t) \not\subset V(F_y) \cup V(F_z)$ nor $V(y, t) \not\subset V(F_x) \cup V(F_z)$ and $V(z, t) \not\subset V(F_x) \cup V(F_y)$. Hence we take $A \in \mathbf{V}^\vee$ such that $\dim V(\tilde{F}_z, \tilde{A}_3) = \dim V(\tilde{A}_1, \tilde{A}_2) = 1$ and $V(\tilde{A}_1) \cup V(\tilde{A}_2) \cup V(\tilde{A}_3)$ contains neither $V(x, t)$ nor $V(y, t)$ nor $V(z, t)$ and such that a, b, c, d are non null and such that $\dim(\mathcal{H}_{c,d} \cap V(\tilde{F}_z, \tilde{A}_3)) = 0$. We consider a generic plane $\mathcal{H}_0 \subset \mathbb{P}^3$ such that $\delta(A) \notin \mathcal{H}_0$, $\mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(\tilde{F}_z, \tilde{A}_3) = \emptyset$, $\dim(\mathcal{H}_0 \cap V(\tilde{A}_1, \tilde{A}_2)) = 0$ and $\mathcal{H}_0 \cap V(\tilde{A}_1, \tilde{A}_2, z, t) = \emptyset$ so that, due to (17), we have

$$\mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(\tilde{A}_1, \tilde{A}_2) = \mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(\tilde{F}_z) \setminus V(t) \quad (18)$$

(since $\delta(A) \notin \mathcal{H}_0$ and since $\mathcal{H}_0 \cap V(\tilde{A}_1, \tilde{A}_2, z, t) = \emptyset$). Hence $\mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap \tilde{\mathcal{P}}_{A,S} = \emptyset$. Following the proof of (10), we obtain that, for a generic plane $\mathcal{H}_0 \subset \mathbb{P}^3$, we have

$$\begin{aligned} \deg \tilde{\mathcal{P}}_{A,S} &= \deg(\tilde{A}_1)^2 - \sum_{P \in \mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(\tilde{A}_1, \tilde{A}_2)} i_P(\mathcal{H}_0, \mathcal{H}_{c,d}, V(\tilde{A}_1, \tilde{A}_2)) \\ &= \deg(\tilde{A}_1)^2 - \sum_{P \in \mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(\tilde{F}_z) \setminus V(t)} i_P(\mathcal{H}_0, \mathcal{H}_{c,d}, V(t\tilde{A}_1, t\tilde{A}_2)) \\ &= \deg(\tilde{A}_1)^2 - \sum_{P \in \mathcal{H}_0 \cap \mathcal{H}_{c,d} \cap V(\tilde{F}_z) \setminus V(t)} i_P(\mathcal{H}_0, \mathcal{H}_{c,d}, V(\tilde{F}_z)) \end{aligned}$$

due to (18), to (13) and since $\delta(A) \notin \mathcal{H}_0$. Hence we have

$$\begin{aligned} \deg \tilde{\mathcal{P}}_{A,S} &= \deg(\tilde{A}_1)^2 - \deg \tilde{F}_z + i_{P_0}(\mathcal{H}_0, \mathcal{H}_{c,d}, V(\tilde{F}_z)) \\ &= \deg(\tilde{A}_1)^2 - \deg \tilde{F}_z + 1, \end{aligned} \tag{19}$$

where P_0 is the point contained in $\mathcal{H}_0 \cap V(z, t)$. Indeed, if $\mathcal{H}_0 = V(\alpha x + \beta y + \gamma z + \delta t)$, a parametrization of $\mathcal{H}_0 \cap \mathcal{H}_{c,d}$ at $P_0[\beta : -\alpha : 0 : 0]$ is given by $\psi(t) = [\beta : -\alpha - (\gamma ct/d) - \delta t : (c\beta t/d) : t\beta]$. Moreover, since $H_1(x, y, z, 0)$ divides $G'(x^2 + y^2 + z^2)$, \tilde{F}_z has the form $\tilde{G}_0(x^2 + y^2 + z^2)z + t\tilde{K}_0(x, y, z, t)$, and so $i_{P_0}(\mathcal{H}_0, V(\tilde{F}_z, ct - dz)) = \text{val}_t \tilde{F}_z(\psi(t)) = \text{val}_t(c\beta t\tilde{G}_0(\beta^2 + \alpha^2)/d + \beta t\tilde{K}_0(\beta, -\alpha, 0, 0)) = 1$, for a generic \mathcal{H}_0 (given a generic $A \in \mathbf{V}^\vee$ such that $c\tilde{G}_0(\beta^2 + \alpha^2) \neq d\tilde{K}_0(\beta, -\alpha, 0, 0)$ in $\mathbb{C}[\alpha, \beta]$).

The remark follows from (15) and (19). \square

Example 12. Observe that the only irreducible quadrics $\mathcal{S} = V(F) \subset \mathbb{P}^3$ such that $\dim \mathcal{B} \geq 2$ are the spheres and cones, i.e. with F of the following form

$$F(x, y, z, t) = (x - x_0 t)^2 + (y - y_0 t)^2 + (z - z_0 t)^2 + a_0 t^2,$$

where x_0, y_0, z_0, a_0 are complex numbers (it is a sphere if $a_0 \neq 0$ and it is a cone otherwise).

Hence, due to Proposition 9, the degree of a generic normal polar of any irreducible surface $\mathcal{S} = V(F) \subset \mathbb{P}^3$ of degree 2 which is neither a sphere nor a cone is 3.

Moreover, for a sphere or for a cone, applying Proposition 11 with $H = t$, $\tilde{\mathcal{P}}_{A,S}$ is a line for a generic $A \in \mathbf{V}^\vee$.

2.4. Geometric study of the base points. We set $\mathcal{B}_\mathcal{S} := \mathcal{B} \cap \mathcal{S}$ for the base points of $\alpha|_\mathcal{S}$. We identify the set $\mathcal{B}_\mathcal{S}$ thanks to the next proposition. We recall that the **umbilical curve** \mathcal{C}_∞ is the set of circular points at infinity, i.e.

$$\mathcal{C}_\infty := \{[x : y : z : t] \in \mathbb{P}^3 : t = 0, x^2 + y^2 + z^2 = 0\}.$$

We also set $\mathcal{S}_\infty := \mathcal{S} \cap \mathcal{H}^\infty$.

Definition 13. We denote by $\text{Sing}(\mathcal{S})$ the set of singular points of \mathcal{S} .

The set of contact at infinity of \mathcal{S} , denoted by $\mathcal{C}_\infty(\mathcal{S})$, is the set of points of \mathcal{S} where the tangent plane is \mathcal{H}_∞ .

The set of umbilical contact of \mathcal{S} , denoted by $\mathcal{U}_\infty(\mathcal{S})$, is the set of points of contact of \mathcal{S}_∞ with the umbilical in \mathcal{H}_∞ .

Proposition 14. The set of base points of $\alpha|_\mathcal{S}$ is $\text{Sing}(\mathcal{S}) \cup \mathcal{C}_\infty(\mathcal{S}) \cup \mathcal{U}_\infty(\mathcal{S})$.

Proof. We have to prove that the base points of $\alpha|_\mathcal{S}$ are the singular points of \mathcal{S} (i.e. the points of $V(F_x, F_y, F_z, F_t)$), the contact points of \mathcal{S} with \mathcal{H}^∞ (i.e. the points $P \in \mathcal{S}$ such that $\mathcal{T}_P \mathcal{S} = \mathcal{H}^\infty$)

and the contact points of \mathcal{S}_∞ with \mathcal{C}_∞ in \mathcal{H}_∞ (i.e. the points $P \in \mathcal{S} \cap \mathcal{C}_\infty$ such that the tangent lines $\mathcal{T}_P(\mathcal{S}_\infty)$ and $\mathcal{T}_P(\mathcal{C}_\infty)$ are the same).

Let $m \in \mathcal{S}$. We have

$$\begin{aligned} m \in \mathcal{B}_\mathcal{S} &\Leftrightarrow \mathbf{m} \wedge \mathbf{n}_\mathcal{S}(\mathbf{m}) = 0 \\ &\Leftrightarrow \mathbf{n}_\mathcal{S}(\mathbf{m}) = \mathbf{0} \text{ or } m = n_\mathcal{S}(m) \\ &\Leftrightarrow m \in V(F_x, F_y, F_z) \text{ or } m = n_\mathcal{S}(m). \end{aligned}$$

Now $m \in V(F_x, F_y, F_z)$ means either that m is a singular point of \mathcal{S} or that $\mathcal{T}_m\mathcal{S} = \mathcal{H}^\infty$.

Let $m = [x : y : z : t] \in \mathcal{S}$ be such that $m = n_\mathcal{S}(m)$. So $[x : y : z : t] = [F_x : F_y : F_z : 0]$. In particular $t = 0$. Due to the Euler identity, we have $0 = xF_x + yF_y + zF_z + tF_t = F_x^2 + F_y^2 + F_z^2$ and so $x^2 + y^2 + z^2 = 0$. Hence m is in the umbilical curve \mathcal{C}_∞ . Observe that the tangent line $\mathcal{T}_m\mathcal{C}_\infty$ to \mathcal{C}_∞ at m has equations $T = 0$ and $\delta^{-1}([x : y : z : 0])$ and that the tangent line $\mathcal{T}_m\mathcal{S}_\infty$ to \mathcal{S}_∞ at m has equations $T = 0$ and $\delta^{-1}(n_\mathcal{S}(m))$. We conclude that $\mathcal{T}_m\mathcal{C}_\infty = \mathcal{T}_m\mathcal{S}_\infty$.

Conversely, if $m = [x : y : z : 0]$ is a nonsingular point of $\mathcal{S}_\infty \cap \mathcal{C}_\infty$ such that $\mathcal{T}_m\mathcal{C}_\infty = \mathcal{T}_m\mathcal{S}_\infty$, then the linear spaces $\text{Span}((x, y, z, t), (0, 0, 0, 1))$ and $\text{Span}((F_x, F_y, F_z, F_t), (0, 0, 0, 1))$ are equal which implies that $[x : y : z : 0] = [F_x : F_y : F_z : 0]$. \square

In other words, the base points $\mathcal{B}_\mathcal{S}$ of $\alpha|_\mathcal{S}$ are the singular points of \mathcal{S} and the common points of \mathcal{S} with its dual surface² at infinity.

Example 15. For the saddle surface $\mathcal{S}_1 = V(xy - zt)$, the set $\mathcal{B}_{\mathcal{S}_1}$ contains a single point $[0 : 0 : 1 : 0]$ which is a point of contact at infinity of \mathcal{S}_1 .

For the ellipsoid $\mathcal{E}_1 := V(x^2 + 2y^2 + 4z^2 - t^2)$, the set $\mathcal{B}_{\mathcal{E}_1}$ is empty.

For the ellipsoid $\mathcal{E}_2 := V(x^2 + 4y^2 + 4z^2 - t^2)$, the set $\mathcal{B}_{\mathcal{E}_2}$ has two elements: $[0 : 1 : \pm i : 0]$ which are umbilical contact points of \mathcal{E}_2 .

2.5. Computation of the normal class of a surface.

Theorem 16 (Normal class formula). Let \mathcal{S} be an irreducible surface of \mathbb{P}^3 of degree $d_\mathcal{S} \geq 2$ such that $\#\mathcal{B}_\mathcal{S} < \infty$. Assume moreover that $F_x F_y F_z \neq 0$ in $\mathbb{C}[x, y, z, t]$. Then, for a generic $A \in \mathbf{V}^\vee$, we have

$$c_\nu(\mathcal{S}) = d_\mathcal{S}^3 - d_\mathcal{S}^2 + d_\mathcal{S} - \sum_{P \in \mathcal{B}_\mathcal{S}} i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}). \quad (20)$$

Proof. We keep the notation $\delta(A) = [a : b : c : d]$. Observe that, for a generic $A \in \mathbf{V}^\vee$, since $\overline{\alpha(\mathcal{S})}$ is irreducible of dimension at most 2, we have $\#\bigcap_{i=1}^4 \mathcal{H}_{A, i} \cap \overline{\alpha(\mathcal{S})} < \infty$ and so $\#\mathcal{P}_{A, \mathcal{S}} \cap \mathcal{S} < \infty$ (since $\mathcal{B}_\mathcal{S} < \infty$).

Since $\dim \mathcal{P}_{A, \mathcal{S}} = 1$ and $\#\mathcal{S} \cap \mathcal{P}_{A, \mathcal{S}} < \infty$ for a generic A in \mathbf{V}^\vee , due to Proposition 9 and to the Bezout formula, we have:

$$\begin{aligned} d_\mathcal{S}(d_\mathcal{S}^2 - d_\mathcal{S} + 1) &= \deg(\mathcal{S} \cap \mathcal{P}_{A, \mathcal{S}}) \\ &= \sum_P i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}) \\ &= \sum_{P \in \mathcal{B}_\mathcal{S}} i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}) + \sum_{P \in \mathcal{S} \setminus \mathcal{B}} i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}). \end{aligned}$$

²We call dual surface of \mathcal{S} its image by the Gauss map.

Now let us prove that, for a generic $A \in \mathbf{V}^\vee$,

$$\sum_{P \in \mathcal{S} \setminus \mathcal{B}} i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}) = \#((\mathcal{S} \cap \mathcal{P}_{A, \mathcal{S}}) \setminus \mathcal{B}). \quad (21)$$

Since α is a rational map, $\overline{\alpha(\mathcal{S})}$ is irreducible and its dimension is at most 2.

Assume first that $\dim \overline{\alpha(\mathcal{S})} < 2$. For a generic $A \in \mathbf{V}^\vee$, the plane $\bigcap_{i=1}^4 \mathcal{H}_{A, i} = \bigcap_{i=1}^4 \psi_i^{-1}(\mathcal{H}_A)$ does not meet $\overline{\alpha(\mathcal{S})}$ and so the left and right hand sides of (21) are both zero. So Formula (20) holds true with $c_\nu(\mathcal{S}) = 0$.

Assume now that $\dim \overline{\alpha(\mathcal{S})} = 2$. Then, for a generic $A \in \mathbf{V}^\vee$, the plane $\bigcap_{i=1}^4 \mathcal{H}_{A, i}$ meets $\overline{\alpha(\mathcal{S})}$ transversally (with intersection number 1 at every intersection point) and does not meet $\overline{\alpha(\mathcal{S})} \setminus \alpha(\mathcal{S})$. This implies that, for a generic $A \in \mathbf{V}^\vee$, we have $i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}) = 1$ for every $P \in (\mathcal{S} \cap \mathcal{P}_{A, \mathcal{S}}) \setminus \mathcal{B}$ and so (21) follows. Hence, for a generic $A \in \mathbf{V}^\vee$, we have

$$\begin{aligned} d_{\mathcal{S}}(d_{\mathcal{S}}^2 - d_{\mathcal{S}} + 1) &= \sum_{P \in \mathcal{B}_{\mathcal{S}}} i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}) + \#\{P \in \mathcal{S} \setminus \mathcal{B} : \delta(A) \in \mathcal{N}_m \mathcal{S}\} \\ &= \sum_{P \in \mathcal{B}_{\mathcal{S}}} i_P(\mathcal{S}, \mathcal{P}_{A, \mathcal{S}}) + c_\nu(\mathcal{S}), \end{aligned}$$

which gives (20). \square

We can now prove Theorem 1.

Corollary 17. *For a generic irreducible surface $\mathcal{S} \subset \mathbb{P}^3$ of degree $d_{\mathcal{S}}$, $\mathcal{B}_{\mathcal{S}} = \emptyset$, $V(F_x F_y F_z) \neq \mathbb{P}^3$ and so*

$$c_\nu(\mathcal{S}) = d_{\mathcal{S}}^3 - d_{\mathcal{S}}^2 + d_{\mathcal{S}}.$$

If $\dim \mathcal{B} = 2$, we saw in Proposition 11 that we can adapt our study to compute the degree of the reduced normal polar $\tilde{\mathcal{P}}_{A, \mathcal{S}}$ associated to the rational map $\tilde{\alpha} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that $\alpha = H \cdot \tilde{\alpha}$. Recall that we have denoted by $\tilde{\mathcal{B}}$ the set of base points of this rational map. Using Proposition 11 and following the proof of Theorem 16, we obtain:

Proposition 18. *Assume that $\dim \mathcal{B} = 2$, with two dimensional part $V(H) \subset \mathbb{P}^3$ for some homogeneous polynomial $H \in \mathbb{C}[x, y, z, t]$ of degree d_H . Assume moreover that $F_x F_y F_z \neq 0$ in $\mathbb{C}[x, y, z, t]$ and that $\#(\mathcal{S} \cap \tilde{\mathcal{B}}) < \infty$. Then, for a generic $A \in \mathbf{V}^\vee$ we have*

$$c_\nu(\mathcal{S}) = d_{\mathcal{S}} \cdot (\tilde{d}_{\mathcal{S}}^2 - \tilde{d}_{\mathcal{S}} + 1) - \sum_{P \in \tilde{\mathcal{B}} \cap \mathcal{S}} i_P(\mathcal{S}, \tilde{\mathcal{P}}_{A, \mathcal{S}}),$$

with $\tilde{d}_{\mathcal{S}} := d_{\mathcal{S}} - d_H$.

3. STUDY OF THE NORMAL CLASS OF QUADRICS

The aim of the present section is the study of the normal class of every irreducible quadric. Let $\mathcal{S} = V(F) \subset \mathbb{P}^3$ be an irreducible quadric. We recall that, up to a composition by $\varphi \in \widehat{\text{Sim}}_{\mathbb{C}}(3)$, one can suppose that F has the one of the following forms:

- (a) $F(x, y, z, t) = x^2 + \alpha y^2 + \beta z^2 + t^2$
- (b) $F(x, y, z, t) = x^2 + \alpha y^2 + \beta z^2$
- (c) $F(x, y, z, t) = x^2 + \alpha y^2 - 2tz$
- (d) $F(x, y, z, t) = x^2 + \alpha y^2 + t^2,$

with α, β two non zero complex numbers. Spheres, ellipsoids and hyperboloids are particular cases of (a), paraboloids (including the saddle surface) are particular cases of (c), (b) correspond to cones and (d) to cylinders.

We will see, in Appendix A, that in the case (d) (cylinders) and in the cases (a) and (b) with $\alpha = \beta = 1$, the normal class of the quadric is naturally related to the normal class of a conic.

Proposition 19. *The normal class of a sphere is 2.*

The normal class of a quadric $V(F)$ with F given by (a) is 6 if $1, \alpha, \beta$ are pairwise distinct.

The normal class of a quadric $V(F)$ with F given by (a) is 4 if $\alpha = 1 \neq \beta$.

The normal class of a quadric $V(F)$ with F given by (b) is 4 if $1, \alpha, \beta$ are pairwise distinct.

The normal class of a quadric $V(F)$ with F given by (b) is 2 if $\alpha = 1 \neq \beta$.

The normal class of a quadric $V(F)$ with F given by (b) is 0 if $\alpha = \beta = 1$.

The normal class of a quadric $V(F)$ with F given by (c) is 5 if $\alpha \neq 1$ and 3 if $\alpha = 1$.

The normal class of a quadric $V(F)$ with F given by (d) is 4 if $\alpha \neq 1$ and 2 if $\alpha = 1$.

Corollary 20. *The normal class of the saddle surface $\mathcal{S}_1 = V(xy - zt)$ is 5.*

The normal class of the ellipsoid $\mathcal{E}_1 = V(x^2 + 2y^2 + 4z^2 - t^2)$ with three different length of axis is 6.

The normal class of the ellipsoid $\mathcal{E}_2 = V(x^2 + 4y^2 + 4z^2 - t^2)$ with two different length of axis is 4.

Proof of Proposition 19. Let $\mathcal{S} = V(F)$ be a quadric with F of the form (a), (b), (c) or (d).

- The easiest cases is (a) with $1, \alpha, \beta$ pairwise distinct since the set of base points $\mathcal{B}_{\mathcal{S}}$ is empty. In this case, since the generic degree of the normal polar curves is 3 and since \mathcal{E}_1 has degree 2, we simply have $c_{\nu}(\mathcal{E}_1) = 2 \cdot 3 = 6$ (due to Theorem 16).
- The case of a sphere \mathcal{S} is analogous. In this case, $\tilde{\mathcal{B}} \cap \mathcal{S} = \emptyset$ and $\deg \tilde{\mathcal{P}}_{A,\mathcal{S}} = 1$ for a generic $A \in \mathbf{V}^{\vee}$ (see Example 12). Hence, we have $c_{\nu}(\mathcal{E}_1) = 2 \cdot 1 = 2$ (due to Proposition 18).

In the other cases, the set of base points on \mathcal{S} will not be empty and we will have to compute intersection multiplicity of the surface with a generic normal polar curve at the base points P . To this end, we compute a parametrization $t \mapsto \psi(t)$ (in a chart $x = 1, y = 1, z = 1$ or $t = 1$) of the normal polar curve $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of each base point P and use the formula $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \text{val}(F \circ \psi)$.

- In case (a) with $\alpha = 1 \neq \beta$, the set $\mathcal{B}_{\mathcal{S}}$ contains two points $[1 : \pm i : 0 : 0]$. We find the parametrization $\psi(y) = [1 : \pm i + y : 0 : 0]$ of $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of $P[1 : \pm i : 0 : 0]$, which gives $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \text{val}_z(1 + (\pm i + y)^2) = 1$ and so $c_{\nu}(\mathcal{S}) = 2 \cdot 3 - 1 - 1 = 4$.
- In case (b) with α, β and 1 are pairwise distinct, the set $\mathcal{B}_{\mathcal{S}}$ contains a single point $P[0 : 0 : 0 : 1]$ and a parametrization of $\mathcal{P}_{A,\mathcal{S}}$ in a neighbourhood of P is

$$\psi(x) = \left[x : -\frac{bx}{d(\alpha - 1)x - a} : \frac{cx}{a + d(1 - \beta)x} : 1 \right]. \quad (22)$$

Hence $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \text{val}_x(F(\psi(x))) = 2$ and so $c_{\nu}(\mathcal{S}) = 2 \cdot 3 - 2 = 4$.

- In case (b) with $\alpha = 1 \neq \beta$, we have $\mathcal{B}_{\mathcal{S}} = \{P, P'_+, P'_-\}$ with $P[0 : 0 : 0 : 1]$ and $P'_{\pm}[1 : \pm i : 0 : 0]$. A parametrization of $\mathcal{P}_{A,\mathcal{S}}$ in a neighbourhood of P is given by (22)

with $\alpha = 1$ and so $i_P(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 2$. A parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at a neighbourhood of P'_\pm is $\psi(z) = [1 : \pm i + y : 0 : 0]$ and so $i_{P'_\pm}(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 1$. Hence $c_\nu(\mathcal{S}) = 2 \cdot 3 - 2 - 1 - 1 = 2$.

- In case (b) with $\alpha = \beta = 1$, for a generic $A \in \mathbf{V}^\vee$, we have $\deg \tilde{\mathcal{P}}_{A,\mathcal{S}} = 1$ (see Example 12) but here $\tilde{\mathcal{B}} \cap \mathcal{S} = \{[0 : 0 : 0 : 1]\}$. We find the parametrization $\psi(x) = [x : (bx/a) : (cx/a) : 1]$ of $\tilde{\mathcal{P}}_{A,\mathcal{S}}$ at the neighbourhood of $P[0 : 0 : 0 : 1]$. Hence $i_P(\mathcal{S}, \tilde{\mathcal{P}}_{A,\mathcal{S}}) = 2$ and so $c_\nu(\mathcal{S}) = 2 \cdot 1 - 2 = 0$.
- In case (c) with $\alpha \neq 1$, the only base point on \mathcal{S} is $P_1[0 : 0 : 1 : 0]$ and a parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of this point is

$$\psi(t) = \left[\frac{at^2}{d + (d-c)t} : \frac{bt^2}{t(d-c\alpha) + \alpha d} : 1 : t \right], \quad (23)$$

which gives $i_{P_1}(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 1$. Hence $c_\nu(\mathcal{S}) = 2 \times 3 - 1 = 5$.

- In case (c) with $\alpha = 1$, there are three base points on \mathcal{S} : $P_1[0 : 0 : 1 : 0]$, $P_{2,\pm}[1 : \pm i : 0 : 0]$. As in the previous case, a parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of P_1 is given by (23) with $\alpha = 1$ and so $i_{P_1}(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = 1$. Now, a parametrization of $\mathcal{P}_{A,\mathcal{S}}$ at the neighbourhood of $P_{2,\pm}$ is $\psi(t) = [1 : \pm i + y : 0 : 0]$ and so $i_{P_{2,\pm}}(\mathcal{S}, \mathcal{P}_{A,\mathcal{S}}) = \text{val}_y(1 + (y \pm i)^2) = 1$.
- For the case (d), due to Proposition 22 (see also Remark 10), $c_\nu(\mathcal{S}) = c_\nu(\mathcal{C})$ with $\mathcal{C} = V(x^2 + \alpha y^2 + z^2) \subset \mathbb{P}^2$ which is a circle if $\alpha = 1$ and an ellipse otherwise. Hence, due to Theorem 2, $c_\nu(\mathcal{C}) = 2 + 2 - 0 - 1 - 1 = 2$ if $\alpha = 1$ and $c_\nu(\mathcal{C}) = 2 + 2 = 4$ otherwise.

□

4. NORMAL CLASS OF A CUBIC SURFACE WITH SINGULARITY E_6

Consider $S = V(F) \subset \mathbb{P}^3$ with $F(x, y, z, t) := x^2z + z^2t + y^3$. \mathcal{S} is a singular cubic surface with E_6 -singularity at $p[0 : 0 : 0 : 1]$. Let a generic $A \in \mathbf{V}^\vee$ with $\delta(A) = [a : b : c : d]$. The ideal of the normal polar $\mathcal{P}_{\mathcal{S},A}$ is given by $I(\mathcal{P}_{\mathcal{S},A}) = \langle H_1, H_2, H_3 \rangle \subset \mathbb{C}[x, y, z, t]$ with $H_1 := (y(x^2 + 2zt) - 3y^2z)d - b(x^2 + 2zt)t + 3y^2ct$, $H_2 := (x(x^2 + 2zt) - 2xz^2)d - a(x^2 + 2zt)t + 2xzt c$ and $H_3 := (-2xzy + 3xy^2)d - 3ay^2t + 2xztb$. We have two base points: p and $q[0 : 0 : 1 : 0]$. Actually q is the contact point of \mathcal{S} with \mathcal{H}_∞ . The curve at infinity $\mathcal{S}_\infty := V(x^2z + y^3) \subset \mathcal{H}_\infty$ has a cusp at q .

(1) Study at p .

Near p the ideal of the normal polar, in the chart $t = 1$, $H_3 = 0$ gives $z = g(x, y) := \frac{3y^2(-xd+a)}{2x(-yd+b)}$. Now $V(A_1(x, y, g(x, y), 1))$ corresponds to a quintic with a cusp at the origine (and with tangent cone $V(y^2)$). Its single branch has Puiseux expansion $y^2 = -\frac{b}{3a}x^3 + o(x^3)$, with probranches $y = \varphi_\varepsilon(x)$ with $\varphi_\varepsilon(x) = i\varepsilon\sqrt{\frac{b}{3a}}x^{\frac{3}{2}} + o(x^{\frac{3}{2}})$ for $\varepsilon \in \{\pm 1\}$. Hence, $g(x, \varphi_\varepsilon(x)) = -\frac{x^2}{2} + o(x^2)$. Hence parametrizations of the probranches of $\mathcal{P}_{A,\mathcal{S}}$ at a neighbourhood of p are

$$\Gamma_\varepsilon(x) = [x : \varphi_\varepsilon(x) : g(x, \varphi_\varepsilon(x)) : 1]$$

and $F(\Gamma_\varepsilon(x)) = -\frac{x^4}{4} + o(x^4)$. Therefore $i_p(\mathcal{P}_{\mathcal{S},A}, \mathcal{S}) = 8$.

(2) Study at q .

Assume that $b = 1$. Near $q[0 : 0 : 1 : 0]$, in the chart $z = 1$, $H_3 = 0$ gives $t = h(x, y) := \frac{d(-2+3y)xy}{3ay^2-2x}$ and $V(H_2(x, y, 1, h(x, y)))$ is a quartic with a (tacnode) double point in $(0, 0)$ with vertical tangent and which has Puiseux expansion

$$x = \theta_\varepsilon(y) = \omega_\varepsilon a y^2 + o(y^2),$$

with $\omega_\varepsilon = \frac{3-d}{2} + \frac{\varepsilon}{2}\sqrt{d(d-6)}$ for $\varepsilon \in \{\pm 1\}$ and $h(\theta_\varepsilon(y), y) = -\frac{2d\omega_\varepsilon}{3-2\omega_\varepsilon}y + o(y)$. Hence parametrizations of the probranches of $\mathcal{P}_{\mathcal{S},A}$ in a neighbourhood of q are given by

$$\Gamma_\varepsilon(y) := [\theta_\varepsilon(y) : y : 1 : h(\theta_\varepsilon(y), y)]$$

for $\varepsilon \in \{\pm 1\}$ and $F(\Gamma_\varepsilon(y)) = -\frac{2\omega_\varepsilon d}{3-2\omega_\varepsilon}y + o(y)$. Hence $i_q(\mathcal{P}_{\mathcal{S},A}, \mathcal{S}) = 2$

Therefore, due to Theorem 16, the normal class of $\mathcal{S} = V(x^2z + z^2t + y^3) \subset \mathbb{P}^3(\mathbb{C})$ is

$$c_\nu(\mathcal{S}) = 3 \cdot (3^2 - 3 + 1) - 8 - 2 = 11.$$

5. NORMAL CLASS OF PLANAR CURVES : PROOF OF THEOREM 2

Let \mathbf{V} be a three dimensional complex linear space and set $\mathbb{P}^2 := \mathbb{P}(\mathbf{V})$ with projective coordinates x, y, z . We denote by $\ell_\infty = V(z)$ the line at infinity. We consider the natural map $\delta : \mathbf{V}^\vee \dashrightarrow \mathbb{P}^2$ given by $\delta(A) = [A(1, 0, 0) : A(0, 1, 0) : A(0, 0, 1)]$.

Let $\mathcal{C} = V(F) \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 2$. For any nonsingular $m[x : y : z] \in \mathcal{C}$ (with coordinates $\mathbf{m} = (x, y, z) \in \mathbb{C}^3$), we write $\mathcal{T}_m\mathcal{C}$ for the tangent line to \mathcal{C} at m . If $\mathcal{T}_m\mathcal{C} \neq \ell_\infty$, then $n_{\mathcal{C}}(m) = [F_x : F_y : 0]$ is well defined in \mathbb{P}^2 and **the projective normal line** $\mathcal{N}_m\mathcal{C}$ to \mathcal{C} at m is the line $(m n_{\mathcal{C}}(m))$ if $n_{\mathcal{C}}(m) \neq m$. Equations of this normal line are then given by $\delta^{-1}(N_{\mathcal{C}}(m))$ where $N_{\mathcal{C}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is the rational map defined by

$$\mathbf{N}_{\mathcal{C}}(\mathbf{m}) := \mathbf{m} \wedge \begin{pmatrix} F_x \\ F_y \\ 0 \end{pmatrix} = \begin{pmatrix} -zF_y(m) \\ zF_x(m) \\ xF_y(m) - yF_x(m) \end{pmatrix}. \quad (24)$$

Lemma 21. *The base points of $(N_{\mathcal{C}})|_{\mathcal{C}}$ are the singular points of \mathcal{C} , the contact points with the line at infinity and the points of $\{I, J\} \cap \mathcal{C}$.*

Proof. A point $m \in \mathcal{C}$ is a base point of $N_{\mathcal{C}}$ if and only if $F_x = F_y = 0$ or $z = xF_y - yF_x = 0$. Hence, singular points of \mathcal{C} are base points of $N_{\mathcal{C}}$.

Let $m = [x : y : z]$ be a nonsingular point of \mathcal{C} . First $F_x = F_y = 0$ is equivalent to $\mathcal{T}_m\mathcal{C} = \ell_\infty$. Assume now that $z = xF_y - yF_x = 0$ and $(F_x, F_y) \neq (0, 0)$. Then $m = [x : y : 0] = [F_x : F_y : 0]$ and, due to the Euler formula, we have $0 = -zF_z = xF_x + yF_y$ and so $x^2 + y^2 = 0$, which implies $m = I$ or $m = J$.

Finally observe that if $m \in \{I, J\} \cap \mathcal{C}$, then $m = [-y : x : 0]$ and, due to the Euler formula, $0 = -zF_z = xF_x + yF_y = xF_y - yF_x$. \square

Hence the set of base points of $(N_{\mathcal{C}})|_{\mathcal{C}}$ is at most finite (since $\mathcal{C} \neq \ell_\infty$).

Since the degree of each non zero coordinate of $\mathbf{N}_{\mathcal{C}}$ is d , due to the fundamental lemma of [5], we have

$$c_\nu(\mathcal{C}) = d^2 - \sum_{P \in \text{Base}((N_{\mathcal{C}})|_{\mathcal{C}})} i_P(\mathcal{C}, V(L \circ \mathbf{N}_{\mathcal{C}})), \quad (25)$$

for a generic $L \in \mathbf{V}^\vee$. The set $V(L \circ \mathbf{N}_{\mathcal{C}}) \subset \mathbb{P}^2$ is called **the normal polar** of \mathcal{C} with respect to L . It satisfies

$$m \in V(L \circ \mathbf{N}_{\mathcal{C}}) \iff \mathbf{N}_{\mathcal{C}}(\mathbf{m}) = 0 \text{ or } \delta(L) \in \mathcal{N}_m(\mathcal{C}).$$

Now, to compute the generic intersection numbers, we use the notion of probranches [4, 9, 10]. See section 4 of [5] for details. Let $P \in \mathcal{C}$ be a base point of $N_{\mathcal{C}}$ and let us write μ_P for the multiplicity of \mathcal{C} at P . Recall that $\mu_P = 1$ means that P is a nonsingular point of \mathcal{C} . Let $M \in GL(\mathbf{V})$ be such that $M(\mathbf{O}) = \mathbf{P}$ with $\mathbf{O} = (0, 0, 1)$ (we set also $O = [0 : 0 : 1]$) and such

that $V(x)$ is not contained in the tangent cone of $V(F \circ M)$ at O . Recall that the equation of this tangent cone is the homogeneous part of lowest degree in (x, y) of $F(x, y, 1) \in \mathbb{C}[x, y]$ and that this lowest degree is μ_P . Using the combination of the Weierstrass preparation theorem and of the Puiseux expansions,

$$F \circ M(x, y, 1) = U(x, y) \prod_{j=1}^{\mu_P} (y - g_j(x)),$$

for some $U(x, y)$ in the ring of convergent series in x, y with $U(0, 0) \neq 0$ and where $g_j(x) = \sum_{m \geq 1} a_{j,m} x^{\frac{m}{q_j}}$ for some integer $q_j \neq 0$. The $y = g_j(x)$ correspond to the equations of the probranches of \mathcal{C} at P . Since $V(x)$ is not contained in the tangent cone of $V(F \circ M)$ at O , the valuation in x of g_j is strictly larger than or equal to 1 and so the probranch $y = g_j(x)$ is tangent to $V(y - xg'_j(0))$. We write $\mathcal{T}_P^{(i)} := M(V(y - xg'_j(0)))$ the associated (eventually singular) tangent line to \mathcal{C} at P ($\mathcal{T}_P^{(i)}$ is the tangent to the branch of \mathcal{C} at P corresponding to this probranch) and we denote by $i_P^{(j)}$ the tangential intersection number of this probranch:

$$i_P^{(j)} = \text{val}_x(g_j(x) - xg'_j(0)) = \text{val}_x(g_j(x) - xg'_j(x)).$$

We recall that for any homogeneous polynomial $H \in \mathbb{C}[x, y, z]$, we have

$$\begin{aligned} i_P(\mathcal{C}, V(H)) &= i_O(V(F \circ M), V(H \circ M)) \\ &= \sum_{j=1}^{\mu_P} \text{val}_x(H(M(G_j(x)))) \end{aligned}$$

where $G_j(x) := (x, g_j(x), 1)$. With these notations and results, we have

$$\Omega(\mathcal{C}, \ell_\infty) = \sum_{P \in \mathcal{C} \cap \ell_\infty} (i_P(\mathcal{C}, \ell_\infty) - \mu_P(\mathcal{C})) = \sum_{P \in \mathcal{C} \cap \ell_\infty} \sum_{j: \mathcal{T}_P^{(j)} = \ell_\infty} (i_P^{(j)} - 1).$$

For a generic $L \in \mathbf{V}^\vee$, we also have

$$\begin{aligned} i_P(\mathcal{C}, V(L \circ N_{\mathcal{C}})) &= \sum_{j=1}^{\mu_P} \text{val}_x(L(N_{\mathcal{C}}(M(G_j(x)))))) \\ &= \sum_{j=1}^{\mu_P} \min_k \text{val}_x([N_{\mathcal{C}} \circ M]_k(G_j(x))), \end{aligned}$$

where $[\cdot]_k$ denotes the k -th coordinate. Moreover, due to (24), as noticed in Proposition 16 of [6], we have

$$N_{\mathcal{C}} \circ M(\mathbf{m}) = \text{Com}(M) \cdot (\mathbf{m} \wedge [\Delta_{\mathbf{A}} G(\mathbf{m}) \cdot \mathbf{A} + \Delta_{\mathbf{B}} G(\mathbf{m}) \cdot \mathbf{B}]),$$

where $G := F \circ M$, $\mathbf{A} := M^{-1}(1, 0, 0)$, $\mathbf{B} := M^{-1}(0, 1, 0)$ and $\Delta_{(x_1, y_1, z_1)} H = x_1 H_x + y_1 H_y + z_1 H_z$. As observed in Lemma 33 of [5], we have

$$\Delta_{(x_1, y_1, z_1)} G(x, g_j(x), 1) = R_j(x) W_{(x_1, y_1, z_1), j}(x),$$

where $R_j(x) = U(x, g_j(x)) \prod_{j' \neq j} (g_{j'}(x) - g_j(x))$ and $W_{(x_1, y_1, z_1), j}(x) := y_1 - x_1 g'_j(x) + z_1 (x g'_j(x) - g_j(x))$. Therefore, for a generic $L \in \mathbf{V}^\vee$, we have

$$i_P(\mathcal{C}, V(L \circ N_{\mathcal{C}})) = V_P + \sum_{j=1}^{\mu_P} \min_k \text{val}_x([G_j(x) \wedge (W_{\mathbf{A}, j}(x) \cdot \mathbf{A} + W_{\mathbf{B}, j}(x) \cdot \mathbf{B})]_k)$$

where $V_P := \sum_{j=1}^{\mu_P} \sum_{j' \neq j} \text{val}(g_{j'} - g_j)$. Now, we write $h_P^{(j)} := \min_k \text{val}_x([G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})]_k)$ and $h_P := \sum_{j=1}^{\mu_P} h_P^{(j)}$. Observe that $V(P) = 0$ if P is a nonsingular point of \mathcal{C} . We recall that, due to Corollary 31 of [5], we have

$$\sum_{P \in \mathcal{C} \cap \text{Base}(N_{\mathcal{C}})} V_P = d(d-1) - d^\vee$$

and so, due to (25), we obtain

$$c_\nu(\mathcal{C}) = d + d^\vee - \sum_{P \in \mathcal{C} \cap \text{Base}(N_{\mathcal{C}})} h_P. \quad (26)$$

Now we have to compute the contribution $h_P^{(j)}$ of each probranch of each $P \in \mathcal{C} \cap \text{Base}(N_{\mathcal{C}})$. We have observed, in Proposition 29 of [5], that we can adapt our choice of M to each probranch (or, to be more precise, to each branch corresponding to the probranch). This fact will be useful in the sequel. In particular, for each probranch, we take M such that $g'_j(0) = 0$ so $G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})$ can be rewritten:

$$\begin{pmatrix} x \\ g_j(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} x_A y_A - (x_A^2 + x_B^2)g'_j(x) + x_B y_B + (z_A x_A + z_B x_B)(xg'_j(x) - g_j(x)) \\ y_A^2 + y_B^2 - (x_A y_A + x_B y_B)g'_j(x) + (z_A y_A + z_B y_B)(xg'_j(x) - g_j(x)) \\ y_A z_A + y_B z_B - (x_A z_A + x_B z_B)g'_j(x) + (z_A^2 + z_B^2)(xg'_j(x) - g_j(x)) \end{pmatrix}. \quad (27)$$

- Assume first that P is a point of \mathcal{C} outside ℓ_∞ . Then for M as above and such that $z_A = z_B = 0$, we have

$$G_j(0) \wedge (W_{\mathbf{A},j}(0) \cdot \mathbf{A} + W_{\mathbf{B},j}(0) \cdot \mathbf{B}) = \begin{pmatrix} -y_A^2 - y_B^2 \\ x_A y_A + x_B y_B \\ 0 \end{pmatrix}$$

which is non null since $(y_A, y_B) \neq (0, 0)$ and since \mathbf{A} and \mathbf{B} are linearly independent. So $h_P^{(j)} = 0$.

- Assume now that $P \in \mathcal{C} \cap \ell_\infty \setminus \{I, J\}$ and $\mathcal{T}_P^{(j)} \neq \ell_\infty$. Then $y_A + iy_B \neq 0$ and $y_A - iy_B \neq 0$ (since $I, J \notin \mathcal{T}_P^{(j)}$) and so $y_A^2 + y_B^2 \neq 0$ which together with (27) implies that $h_P^{(j)} = 0$ as in the previous case.
- Assume that $P \in \mathcal{C} \cap \ell_\infty \setminus \{I, J\}$ and $\mathcal{T}_P^{(i)} = \ell_\infty$. Assume that $M(1, 0, 0) = (1, i, 0)$. Hence $\mathbf{A} + i\mathbf{B} = (1, 0, 0)$. Then $y_A = y_B = 0$, $x_A + ix_B = 1$, $z_A + iz_B = 0$. So $z_A^2 + z_B^2 = 0$ and $z_A x_A + z_B x_B = z_A \neq 0$ (since $z_B = iz_A$ and $x_B = i(x_A - 1)$). Observe that $P \neq J$ implies also that $x_A - ix_B \neq 0$. So that $x_A^2 + x_B^2 \neq 0$. Hence, due to (27), $G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})$ is equal to

$$\begin{pmatrix} x \\ g_j(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} (x_A^2 + x_B^2)g'_j(x) + z_A(xg'_j(x) - g_j(x)) \\ 0 \\ z_A g'_j(x) \end{pmatrix}.$$

Therefore we have $h_P^{(j)} = \text{val}_x((x_A^2 + x_B^2)g'_j(x)) = i_P^{(j)} - 1$.

- Assume that $P = I$ and that $\mathcal{T}_P^{(j)} = \ell_\infty$. Take M such that $M(\mathbf{O}) = (1, i, 0)$, $\mathbf{B} = (1, 0, 0)$ and so $\mathbf{A} = (-i, 0, 1)$. Due to (27), $G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})$ is equal to

$$\begin{pmatrix} x \\ g_j(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -i(xg'_j(x) - g_j(x)) \\ 0 \\ ig'_j(x) + (xg'_j(x) - g_j(x)) \end{pmatrix}. \quad (28)$$

Now observe that each coordinate has valuation at least equal to $i_P^{(j)} = \text{val } g_j$ and that the term of degree $i_P^{(j)}$ of the second coordinate is the term of degree $i_P^{(j)}$ of

$$-i(xg_j'(x) - g_j(x)) + xig_j'(x) = ig_j(x) \neq 0$$

which is non null. Therefore $h_P^{(j)} = i_P^{(j)}$.

- Assume finally that $P = I$ and that $\mathcal{T}_P^{(j)} \neq \ell_\infty$. Take M such that $M(\mathbf{O}) = (1, i, 0)$, $\mathbf{B} = (0, 1, 0)$ and so $\mathbf{A} = (0, -i, 1)$. Due to (27), $G_j(x) \wedge (W_{\mathbf{A},j}(x) \cdot \mathbf{A} + W_{\mathbf{B},j}(x) \cdot \mathbf{B})$ is equal to

$$\begin{pmatrix} x \\ g_j(x) \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ -i(xg_j'(x) - g_j(x)) \\ -i + (xg_j'(x) - g_j(x)) \end{pmatrix}. \quad (29)$$

Now observe that each coordinate has valuation at least equal to 1 and that the term of degree 1 of the second coordinate is $ix \neq 0$. Hence $i_P^{(j)} = 1$.

Observe that the case $P = J$ can be treated in the same way than the case $P = I$.

Theorem 2 follows from (26) and from the previous computation of h_P .

APPENDIX A. NORMAL CLASS OF SURFACES LINKED TO NORMAL CLASS OF PLANAR CURVES

We consider here two particular cases of surfaces the normal class of which is naturally related to the normal class of a curve.

We start with the case of cylinders.

We call **cylinder of base** $\mathcal{C} = V(G) \subset \mathbb{P}^2$ **and of axis** $V(x, y) \subset \mathbb{P}^3$ the surface $V(F) \subset \mathbb{P}^3$, with $F(x, y, z, t) := G(x, y, t)$.

Proposition 22. *Let $\mathcal{S} = V(F) \subset \mathbb{P}^3$ be a cylinder of axis $V(x, y) \subset \mathbb{P}^3$ and of base $\mathcal{C} = V(G) \subset \mathbb{P}^2$. Then*

$$c_\nu(\mathcal{S}) = c_\nu(\mathcal{C}).$$

Proof. The normal line of \mathcal{S} at $m[x_1 : y_1 : z_1 : t_1] \in \mathcal{S}$ is contained in the plane $V(t_1z - z_1t)$. Hence, for any $P[x_0 : y_0 : z_0 : 1] \in \mathbb{P}^3 \setminus \mathcal{H}_\infty$, we have the following equivalences:

$$\begin{aligned} P \in \mathcal{N}_m \mathcal{S} &\Leftrightarrow P \in (m, n_{\mathcal{S}}(m)) \\ &\Leftrightarrow P \in (m, n_{\mathcal{S}}(m)), \quad m \notin \mathcal{H}_\infty \\ &\Leftrightarrow P \in (m, n_{\mathcal{S}}(m)), \quad t_1 \neq 0, \quad z_1 = z_0 t_1 \\ &\Leftrightarrow t_1 \neq 0, \quad z_1 = z_0 t_1, \quad P_1 \in (m_1, n_{\mathcal{C}}(m_1)) \\ &\Leftrightarrow t_1 \neq 0, \quad z_1 = z_0 t_1, \quad P_1 \in \mathcal{N}_{m_1} \mathcal{C}, \end{aligned}$$

with $m_1[x_1 : y_1 : t_1]$ and $P_1[x_0 : y_0 : 1]$ in \mathbb{P}^2 . Hence $\#\{m \in \mathcal{S} : P \in \mathcal{N}_m \mathcal{S}\} = \#\{m_1 \in \mathcal{C} : P_1 \in \mathcal{N}_{m_1} \mathcal{C}\}$ and the result follows. \square

Another case when the normal class of a surface corresponds to the normal class of a planar surface is the case of revolution surfaces.

We call **algebraic surface of revolution of axis** $V(x, y)$ a surface $V(F) \subset \mathbb{P}^3$, where $F(x, y, z, t) := G(x^2 + y^2, z, t)$, where $G(u, v, w) \in \mathbb{C}[u, v, w]$ is such that $G(u^2, v, w)$ is homogeneous.

Proposition 23. *Let $\mathcal{S} = V(F) \subset \mathbb{P}^3$ be an irreducible surface of degree $d_{\mathcal{S}} \geq 2$. We suppose that \mathcal{S} is an algebraic surface of revolution of axis $V(x, y)$, then $c_{\nu}(\mathcal{S}) = c_{\nu}(\mathcal{C})$ where $\mathcal{C} = V(G(x^2, y, z)) \subset \mathbb{P}^2$ where $G \in \mathbb{C}[u, v, w]$ is such that $F(x, y, z, t) = G(x^2 + y^2, z, t)$.*

Proof. We observe that, for every $m[x_1 : y_1 : z_1 : t_1] \in \mathcal{S}$, we have

$$n_{\mathcal{S}}(m) = [F_x(\mathbf{m}) : F_y(\mathbf{m}) : F_z(\mathbf{m}) : 0] = [2x_1G_u(\mathbf{m}_3) : 2y_1G_u(\mathbf{m}_3) : G_v(\mathbf{m}_3) : 0],$$

with $\mathbf{m}_3 = (x_1^2 + y_1^2, z_1, t_1)$. The normal line of \mathcal{S} at $m[x_1 : y_1 : z_1 : t_1] \in \mathcal{S}$ is contained in the plane $V(x_1y - y_1x)$ if $(x_1, y_1) \neq 0$. Hence, for any $P[x_0 : y_0 : z_0 : 1] \in \mathbb{P}^3 \setminus \mathcal{H}_{\infty}$ with $x_0^2 + y_0^2 \neq 0$ and $x_0 \neq 0$, we have the following equivalences:

$$\begin{aligned} P \in \mathcal{N}_m \mathcal{S} &\Leftrightarrow P \in (m, n_{\mathcal{S}}(m)), \quad m \notin \mathcal{H}_{\infty} \\ &\Leftrightarrow P \in (m, n_{\mathcal{S}}(m)), \quad t_1 \neq 0, \quad x_1y_0 = x_0y_1. \end{aligned}$$

We observe that $x_1y_0 = x_0y_1$ implies that $x_1^2 + y_1^2 \neq 0$ or $(x_1, y_1) = 0$. Hence we have

$$P \in \mathcal{N}_{\mathcal{S}}(m) \Leftrightarrow t_1 \neq 0, \quad y_1 = \frac{x_1y_0}{x_0}, \quad P_1 \in (m_1, N_1),$$

with $P_1[x_0 : z_0 : 1]$, $m_1[x_1 : z_1 : t_1]$ and $N_1 = [2x_1G_u(\mathbf{m}_3) : G_z(\mathbf{m}_3) : 0]$ in \mathbb{P}^2 . Now let δ be a complex number such that $\delta^2x_0^2 = x_0^2 + y_0^2$. We then have

$$\begin{aligned} N_1 \left[\frac{2}{\delta}(x_1\delta)G_u((x_1\delta)^2, z_1, t_1) : G_v((x_1\delta)^2, z_1, t_1) : 0 \right], \\ m_1 \left[\frac{1}{\delta}x_1\delta : z_1 : t_1 \right] \quad \text{and} \quad P_1 \left[\frac{1}{\delta}x_0\delta : z_0 : 1 \right]. \end{aligned}$$

Now we set $m_2[x_1\delta : z_1 : t_1]$ and $P_2[x_0\delta : y_0 : 1]$. We observe that

$$n_{\mathcal{C}}(m_2) [2(x_1\delta)G_u((x_1\delta)^2, z_1, t_1) : G_v((x_1\delta)^2, z_1, t_1) : 0],$$

so that

$$\begin{aligned} P \in \mathcal{N}_m \mathcal{S} &\Leftrightarrow t_1 \neq 0, \quad y_1 = \frac{x_1y_0}{x_0}, \quad P_2 \in (m_2, n_{\mathcal{C}}(m_2)) \\ &\Leftrightarrow t_1 \neq 0, \quad z_1 = z_0t_1, \quad P_2 \in \mathcal{N}_{m_2} \mathcal{C}. \end{aligned}$$

Hence $\#\{m \in \mathcal{S} : P \in \mathcal{N}_{\mathcal{S}}(m)\} = \#\{m_2 \in \mathcal{C} : P_2 \in \mathcal{N}_{\mathcal{C}}(m_2)\}$ and this number does not depend on the choice of δ such that $\delta^2x_0^2 = x_0^2 + y_0^2$. The result follows. \square

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